NGMV Control Using Unstable State-Dependent Multivariable Models

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Abstract

A *Nonlinear Generalized Minimum Variance* control law is derived for systems represented by an input-output state dependent nonlinear subsystem that may be open-loop unstable. The solution is obtained using a model for the multivariable discrete-time process that includes a state-dependent (nonlinear and possibly unstable) model that links the output and any *unstructured* nonlinear input subsystem. The input subsystem can involve an operator of a very general nonlinear form, but this has to be assumed to be stable. This is the first *NGMV* control solution that is suitable for systems containing an unstable *nonlinear* sub-system which is contained in the state-dependent model.

The process is also assumed to include explicit common delays in input or output channels. The generalised minimum variance cost index to be minimised involves both error and control signal costing terms but to increase generality weighted states are also included in the cost index. The controller derived is simple to implement considering the complexity of the system represented. If the plant is stable the controller structure can be manipulated into an internal model control form. This form of the controller is like a nonlinear version of the Smith predictor which is valuable to provide confidence in the solution.

Key words: State-Dependent, Minimum Variance, Optimal Control, Nonlinear, Optimization

1.0 Introduction

The effect of nonlinearities can be very destabilising since they are operating point dependent and they can also introduce mysterious behaviour for those more familiar with essentially linear systems. Hysteresis, for example, is found in many valves and magnetic systems and is not only destabilising but limits tracking performance. Over the last decade nonlinear predictive control laws have been proposed for such systems and these may be very useful for high value important loops but they are not a replacement for say existing *PID* controls where improved performance is desired. This realization has motivated the development of controllers which have a more classical structure, that can be linked to *PID* control and *Smith predictors* but offer a more rigorous scientific basis.

The *MV* controller was derived by *Åström* (1979 [1]), assuming the plant was linear and minimum-phase, and was later derived for processes that could be non-minimum phase. *Hastings-James* (1970 [2]) and later *Clarke and Hastings-James* (1971 [3]) modified the first of these control laws by adding a control signal costing term. This was termed a *Generalized Minimum Variance* (*GMV*) control law and enabled non-minimum phase processes to be stabilized.

A family of so-called *Nonlinear Generalized Minimum Variance* (*NGMV*) controllers was derived more recently for nonlinear *model based* multivariable systems. The assumption was made that the plant model could be decomposed into a set of delay terms, a very general input nonlinear subsystem that had to be stable and a linear subsystem that could be represented in polynomial matrix or state equation form and include unstable modes. This problem was analysed by *Grimble* (2004 [5], 2005 [6]) and *Grimble and Majecki* (2004 [7], 2006 [8]). Unfortunately these solutions did not allow for systems that had nonlinear subsystems that were open loop unstable (the linear subsystems could be unstable) and nonlinearities were only allowed on input subsystems.

The new control law provides for a much more general system description. The nonlinear (*NL*) system model includes subsystems that provide alternative ways of modelling the nonlinearities. The addition of a *state-dependent* model subsystem that may be unstable and follows the more general input operator model can be used for representing nonlinear output subsystems such as sensor nonlinearities.

The main objective is to introduce a *GMV* controller for *nonlinear multivariable processes* that is flexible. The criterion is chosen to include dynamic weightings to enable frequency response shaping but also to ensure a relatively simple controller is obtained. When the system is linear the results revert to those for the usual *GMV* controller.

The input plant sub-system can be in a general nonlinear operator form, which might involve, transfer-operators, neural networks or nonlinear function look-up tables. The state-dependent model links to the system outputs and has the advantage that a prediction equation is easily obtained for this part of the nonlinear system. The solution of the *NGMV* control problem for a system with linear state-space models has been considered in [9] but the use of a *state-dependent model* that may be open-loop unstable, is new. There are no restrictions regarding the stability or minimum-phase properties of the linear or state-dependent plant blocks but no structure is assumed known for the input subsystem and hence this must be assumed to be stable.

The solution for the state-space based version of the *NGMV* optimal control law is relatively simple. For linear systems stability is ensured in *GMV* control when the combination of a control weighting and error weighted plant model is strictly minimum-phase. For nonlinear systems it is shown that a related nonlinear operator must have a stable inverse. A result is obtained that guarantees the existence of at least one set of weights that ensures the existence of this inverse. That is, if say a *PID* controller exists (or classical controller), that will stabilize the nonlinear system without transport delay elements, then it can be shown that a set of cost weightings can be defined to guarantee the existence of this inverse and thereby ensure the stability of the closed-loop (*Khalil* 2002 [22]).

2.0 Nonlinear Operator and State-Dependent Models

The system, shown in Fig. 1, includes the *input nonlinear* plant model and the *output state-dependent* plant model. The system is further augmented by stochastic models for the reference and disturbance signals. The output of the unstructured nonlinear system is denoted $u_1(t-k)$ and this acts as an input to the state-dependent block. The first sub-system is of a general nonlinear operator form $\mathcal{W}_1(.,.)$ where the model without explicit delay $\mathcal{W}_{1k}(.,.)$ and $u_1(t) = (\mathcal{W}_{1k}u)(t)$. The *state-dependent nonlinear model* may be represented by the input-output operator $\mathcal{W}_0(.,.)$, or without delay $\mathcal{W}_{0k}(.,.)$. The total output from the state-dependent block, when disturbances are null, may be written as $y(t) = (\mathcal{W}u)(t) = (\mathcal{W}_{0k}\mathcal{W}_1 u)(t)$.

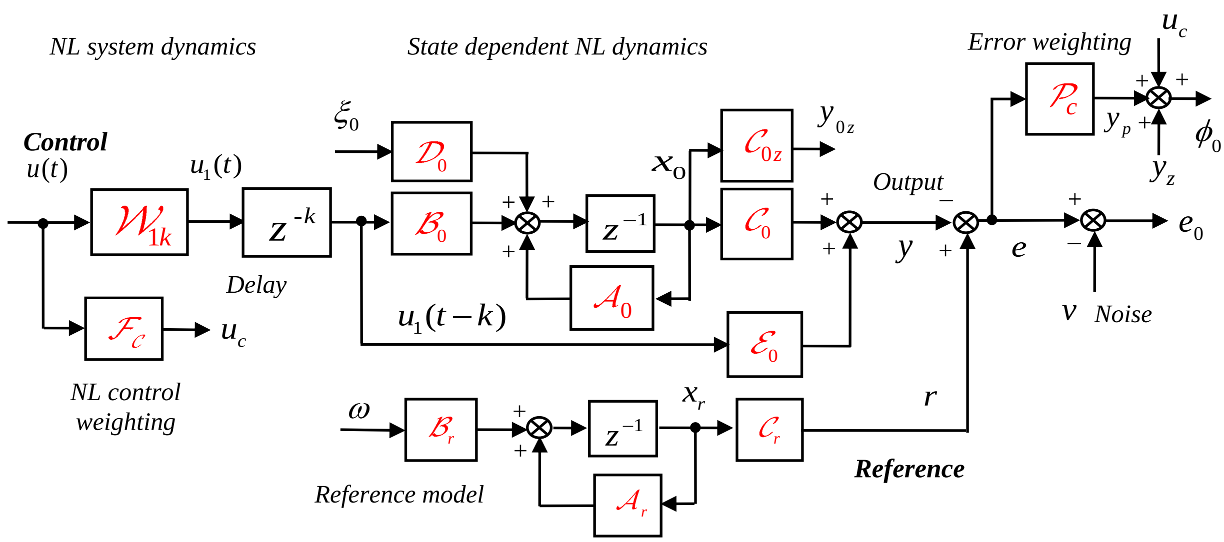

Fig. 1: **Nonlinear Unstructured and State-Dependent Plant Model Including Disturbance and Reference Models** $(u_1(t) = \mathcal{W}_{1k}(.,.)u(t))$

The zero-mean white *measurement noise* is denoted $\{v(t)\}$ and it has a covariance matrix R_f. There is no loss of generality in assuming that the zero-mean, white noise signals: $\{\omega(t)\}$ and $\{\xi_0(t)\}$ that feed the reference and disturbance models, have identity covariance matrices and a Gaussian distribution. It will be shown later that the special structure of the system leads to a prediction equation, which utilises a time-varying *Kalman filter* (*Grimble and Johnson* 1988 [25]). The signals shown in the system model of Fig. 1 may be listed in terms of nonlinear input-output operators as follows:

Error signal: $$e(t) = r(t) - y(t) \quad (1)$$

Reference signal: $$r(t) = \mathcal{W}_r\,\omega(t) \quad (2)$$

Observations signal: $$z(t) = y(t) + v(t) \quad (3)$$

Noisy error signal: $$e_0(t) = r(t) - z(t) \quad (4)$$

The time-functions can be considered to be contained in extensions of the *discrete Marcinkiewicz space* $m_2(R_+, R^n)$ ([12], [13]). This is the space of time sequences with time averaged, square summable signals, which have finite power. The second *nonlinear* subsystem has a well-defined structure and is represented by the state-dependent model equations, shown in Fig.1, introduced in the next section.

2.1 Unstructured and Structured NL Plant Sub-Systems

The nonlinear system models will now be defined (*Slotine and Li*, 1991 [11]). Note that the system is assumed to be in operation from time $t \to -\infty$ and hence initial condition effects will not need to be considered in the theory. In practice there will be a need to initialize the controller but this is discussed later.

Total Nonlinear Plant Model:

$$(\mathcal{W}u)(t) = \left(\mathcal{W}_k\, z^{-k}u\right)(t) \quad (5)$$

Here k denotes the magnitude of the common delay elements in the output signal paths. The delay free input/output model may be written as: $(\mathcal{W}_k u)(t)$ $= (\mathcal{W}_{0k}\mathcal{W}_{1k}\, u)(t)$ and the total forward path *NL* plant model:

$$(\mathcal{W}u)(t) = (\mathcal{W}_{0k}\mathcal{W}_1 u)(t) = \left(\mathcal{W}_{0k}\, z^{-k}\mathcal{W}_{1k}u\right)(t) \quad (6)$$

The signal input to the nonlinear state dependent dynamics is denoted as: $u_1(t) = \mathcal{W}_{1k}(.,.)u(t)$. The nonlinear subsystem: $\mathcal{W}_{1k}$ is assumed to be *finite gain stable* but the nonlinear *state-dependent model* $\mathcal{W}_{0k}$ may be unstable. This latter assumption provides a useful generalisation of the *NGMV approach*, since it is the first time open-loop unstable nonlinear components could be included.

The second nonlinear plant model $\mathcal{W}_{0k}$ is represented in the so-called *state-dependent* state-space form. This has been used for *state-dependent Riccati equation* optimal control solutions (*Hammett* 1997 [14]) and involves state-equation matrices that are time-varying since they are allowed to depend upon the system states. It will also be used for the *reference* and *cost-function weighting* models shown in Fig. 1. Such a model clearly includes *Linear Parameter Varying* (*LPV*) systems ([18]-[20]) where the state models are parameters or input signal dependent. There may of course be some difficulty in approximating a nonlinear system with a state-dependent plus input operator version but the process is very similar to that needed in deriving *LPV* models which have shown considerable promise in applications ranging from automotive to wind turbines (see *Geering* 2005, [27]).

Reference and Weighting State Equation Model: The reference and the error weighting models have the state-space system models, for the ($r \times m$) multivariable system that may be listed as follows:

Reference model

$$x_r(t+1) = \mathcal{A}_r x_r(t) + \mathcal{D}_r\omega(t),\ x_r(t) \in R^{n_r} \quad (7)$$

$$r(t) = \mathcal{C}_r x_r(t) \text{ and } \mathcal{W}_r(z^{-1}) = \mathcal{C}_r(zI - \mathcal{A}_r)^{-1}\mathcal{D}_r \quad (8)$$

Also introduce the *dynamic* error weighting $\mathcal{P}_c(z^{-1})$ that is used in the cost-index defined later. This gives the weighted error signal $y_p(t) = \mathcal{P}_c(z^{-1})\left(r(t) - y(t)\right)$ that has the following state-space representation:

Error weighting

$$x_p(t+1) = \mathcal{A}_p x_p(t) + \mathcal{B}_p\left(r(t) - y(t)\right),\ x_p(t) \in R^{n_p} \quad (9)$$

$$y_p(t) = \mathcal{C}_p x_p(t) + \mathcal{E}_p\left(r(t) - y(t)\right) \quad (10)$$

Combining the reference and error equations for the above sub-systems obtain the state equations by augmenting the state vector as: $x_1(t) = \left[x_r^T(t) \quad x_p^T(t)\right]^T$. Noting equations (7) and (9) the augmented equations may be written as:

$$x_1(t+1) = \mathcal{A}_1 x_1(t) + \mathcal{B}_1 y(t) + \mathcal{D}_1\omega(t) \quad (11)$$

where

$$\mathcal{A}_1 = \begin{bmatrix} \mathcal{A}_r & 0 \\ \mathcal{B}_p\mathcal{C}_r & \mathcal{A}_p \end{bmatrix}, \quad \mathcal{B}_1 = \begin{bmatrix} 0 \\ -\mathcal{B}_p \end{bmatrix}, \quad \mathcal{D}_1 = \begin{bmatrix} \mathcal{D}_r \\ 0 \end{bmatrix} \quad (12)$$

Plant model: A slight extension of this idea is to allow these matrices to be functions of the model input at time *t - k*. The nonlinear plant output sub-system model is therefore assumed to have the following state-dependent model form (*Cloutier et. al.*, 1997 [15], 1998 [16]):

$$\begin{aligned} x_0(t+1) &= \mathcal{A}_0(x_0,u_1)\ x_0(t) \\ &+ \mathcal{B}_0(x_0,u_1)\ u_1(t-k) + \mathcal{D}_0(x_0,u_1)\ \xi_0(t) \end{aligned} \quad (13)$$

$$y(t) = \mathcal{C}_0(x_0,u_1)\,x_0(t) + \mathcal{E}_0(x_0,u_1)\ u_1(t-k) \quad (14)$$

where $x_0(t)$ is a vector of *sub-system* states, $u_1(t)$ is a vector of the *state-dependent* sub-system inputs and $y(t)$ is a vector of output signals. To simplify the notation in (13), (14) write $\mathcal{A}_0(t)$ $= \mathcal{A}_0(x_0(t), u_1(t-k))$ and similarly for the matrices: $\mathcal{B}_0$, $\mathcal{C}_0$ and $\mathcal{E}_0$. Substituting from equations (14) into (11):

$$x_1(t+1) = \mathcal{A}_1 x_1(t) + \mathcal{B}_1\mathcal{C}_0\, x_0(t) + \mathcal{B}_1\mathcal{E}_0\, u_1(t-k) + \mathcal{D}_1\omega(t) \quad (15)$$

Note that the *weighted output* in equation (10) may now be written as:

$$\begin{aligned} y_p(t) &= \mathcal{C}_p x_p(t) + \mathcal{E}_p\left(\mathcal{C}_r x_r(t) - y(t)\right) \\ &= \mathcal{C}_p x_p(t) + \mathcal{E}_p\left(\mathcal{C}_r x_r(t) - (\mathcal{C}_0 x_0(t) + \mathcal{E}_0\, u_1(t-k))\right) \\ &= \mathcal{C}_{0p}\ x_0(t) + \mathcal{C}_{1p}x_1(t) + (\mathcal{E}_\phi\, u_1)(t-k) \end{aligned} \quad (16)$$

where $\mathcal{C}_{0p} = -\mathcal{E}_p\mathcal{C}_0$ and $\mathcal{C}_{1p} = \left[\mathcal{E}_p\mathcal{C}_r \quad \mathcal{C}_p\right]$ and the through term $\mathcal{E}_\phi = -\mathcal{E}_p\mathcal{E}_0$. (17)

2.2 *Combined State Dependent Nonlinear Models*

Let the total combined vector of the state-dependent sub-system model states be defined to have the form $x(t) = [x_0(t)^T \quad x_1(t)^T]^T$. Thence, the *combined state vector* for the nonlinear sub-systems and the related disturbance vector inputs become:

Total NL System State Model:

$$x(t+1) = \mathcal{A}x(t) + \mathcal{B}u_1(t-k) + \mathcal{D}\xi(t) \quad (18)$$

Total NL System Output Model:

$$y(t) = \mathcal{C}x(t) + \mathcal{E}u_1(t-k) \quad (19)$$

Weighted Plant Error Model:

$$y_p(t) = \mathcal{C}_p x(t) + (\mathcal{E}_\phi\, u_1)(t-k) \quad (20)$$

Clearly from equations (11), (13) to (16) the combined state-dependent system models have the form:

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_0 & 0 \\ \mathcal{B}_1\mathcal{C}_0 & \mathcal{A}_1 \end{bmatrix},\ \mathcal{B} = \begin{bmatrix} \mathcal{B}_0 \\ \mathcal{B}_1\mathcal{E}_0 \end{bmatrix},\ \mathcal{E} = \mathcal{E}_0\,,$$

$$\mathcal{D} = \begin{bmatrix} \mathcal{D}_0 & 0 \\ 0 & \mathcal{D}_1 \end{bmatrix}, \mathcal{C} = \begin{bmatrix} \mathcal{C}_0 & 0 \end{bmatrix},\ \mathcal{C}_p = \begin{bmatrix} \mathcal{C}_{0p} & \mathcal{C}_{1p} \end{bmatrix}$$

where the vectors of signals:

$$x(t) = \begin{bmatrix} x_0(t) \\ x_1(t) \end{bmatrix}, \quad \xi(t) = \begin{bmatrix} \xi_0(t) \\ \omega(t) \end{bmatrix} \quad (21)$$

For later use define the *resolvent operator* for the total state-dependent augmented system as:

$$\Phi = (zI - \mathcal{A}(t))^{-1} = (I - \mathcal{A}(t-1)z^{-1})^{-1} z^{-1} \quad (22)$$

2.3 Future Plant Outputs and States

The expression for a k_0 *steps-ahead* state-vector, where $k_0 \geq k$, may be obtained by generalising the above:

$$x(t+k_0) = \mathcal{A}(t+k_0-1)\mathcal{A}(t+k_0-2)....\mathcal{A}(t)x(t)$$
$$+\mathcal{A}(t+k_0-1)\mathcal{A}(t+k_0-2)....\mathcal{A}(t+1)\mathcal{B}(t)u_1(t-k)$$
$$+ .... + \mathcal{A}(t+k_0-1)\mathcal{B}(t+k_0-2)u_1(t+k_0-k-2)$$
$$+\mathcal{B}(t+k_0-1)u_1(t+k_0-k-1)$$
$$+\mathcal{A}(t+k_0-1)\mathcal{A}(t+k_0-2)...\mathcal{A}(t+1)\mathcal{D}(t)\xi(t)$$
$$+ ... + \mathcal{A}(t+k_0-1)\mathcal{D}(t+k_0-2)\xi(t+k_0-2)$$
$$+\mathcal{D}(t+k_0-1)\xi(t+k_0-1) \quad (23)$$

Normally the cost function is defined so that $k_0 = k$ and values of the control signal up to time *t-1* are therefore known, and the future values of these matrices may be computed. These equations may be simplified by defining the product of functions at decreasing times:

$$\coprod_{j=m}^{k} a(t+j) = a(t+k)a(t+k-1)...a(t+m)$$

The k_0 *-steps* prediction of the state and output signals will therefore be defined from the relationships:

$$\hat{x}(t+k_0 \mid t) = \coprod_{j=0}^{k_0-1} \mathcal{A}(t+j)\ \hat{x}(t \mid t)$$
$$+ \coprod_{j=1}^{k_0-1} \mathcal{A}(t+j)\ \mathcal{B}(t)u_1(t-k) + ...$$
$$.... + \coprod_{j=k_0-1}^{k_0-1} \mathcal{A}(t+j)\mathcal{B}(t+j-1)u_1(t+k_0-k-2)$$
$$+\mathcal{B}(t+k_0-1)u_1(t+k_0-k-1) \quad (24)$$

and

$$\hat{y}(t+k_0 \mid t) = \mathcal{C}(t+k_0)\hat{x}(t+k_0 \mid t)$$
$$+\mathcal{E}(t+k_0)u_1(t+k_0-k) \quad (25)$$

These equations may be simplified further, using a f*inite pulse response model*, defined in the form:

$$\mathcal{T}(k_0, z^{-1}) = \coprod_{j=1}^{k_0-1} \mathcal{A}(t+j)\ \mathcal{B}(t)z^{-k} + ...$$
$$+\coprod_{j=k_0-1}^{k_0-1} \mathcal{A}(t+j)\ \mathcal{B}(t+j-1)z^{k_0-k-2}$$
$$+\mathcal{B}(t+k_0-1)z^{k_0-k-1} \quad (26)$$

and by introducing the following simplified notation:

$$\mathcal{A}^{k_0} = \coprod_{j=0}^{k_0-1} \mathcal{A}(t+j) \quad (27)$$

obtain:

$$\hat{x}(t+k_0 \mid t) = \mathcal{A}^{k_0}\ \hat{x}(t \mid t) + \mathcal{T}(k_0, z^{-1})u_1(t) \quad (28)$$

$$\hat{y}(t+k_0 \mid t) = \mathcal{C}(t+k_0)\mathcal{A}^{k_0}\ \hat{x}(t \mid t)$$
$$+\left(\mathcal{C}(t+k_0)\mathcal{T}(k_0, z^{-1})u_1(t) + \mathcal{E}(t+k_0)u_1(t+k_0-k)\right)$$
$$= \mathcal{C}(t+k_0)\mathcal{A}^{k_0}\ \hat{x}(t \mid t)$$
$$+\left(\mathcal{C}(t+k_0)\mathcal{T}(k_0, z^{-1}) + \mathcal{E}(t+k_0)z^{k_0-k}\right)u_1(t) \quad (29)$$

The total vector of state-estimates for the state-dependent subsystems can be written as:

$$\hat{x}(t+j \mid t) = \begin{bmatrix} \hat{x}_0(t+j \mid t) \\ \hat{x}_1(t+j \mid t) \end{bmatrix} \quad \text{for} \quad j \geq 1 \quad (30)$$

3.0 Kalman Predictor for the State-Dependent System

The *Kalman filter* equations introduced below are well known (*Anderson and Moore*, 1979 [24]). The result below is extended in an obvious way to accommodate the *delays on input channels* and *through terms.* A modification is also made to the *Kalman filter* equations to allow for a known bias signal on the measurements. Note that such changes, do not affect the basic stochastic relationships, or the gain of the optimal filter. Recall the total system may be represented in the following state-dependent form that is similar to a known time-varying linear system if the past values of states are for the moment assumed known.

Total Plant Model:

$$x(t+1) = \mathcal{A}x(t) + \mathcal{B}u_1(t-k) + \mathcal{D}\xi(t) \quad (31)$$

Plant Output Signal

$$y(t) = \mathcal{C}x(t) + \mathcal{E}u_1(t-k) \quad (32)$$

Observations Signal:

$$z(t) = y(t) + v(t) = \mathcal{C}x(t) + \mathcal{E}u_1(t-k) + v(t) \quad (33)$$

Controller Input:

$$e_0(t) = r(t) - z(t) = r(t) - \left(\mathcal{C}x(t) + \mathcal{E}u_1(t-k) + v(t)\right)$$
$$= \mathcal{C}_e x(t) - \mathcal{E}u_1(t-k) - v(t) \quad (34)$$

where $x(t) \in R^n$ and $\mathcal{C}_e = \begin{bmatrix} -\mathcal{C}_0 & \mathcal{C}_r & 0 \end{bmatrix}$ denotes the output map taken from the total system states to the error channel. Note that the controller is one degree of freedom so that the *Kalman filter* is driven from the noisy error signal channel. Clearly the state-vector may be written as:

$$x(t) = (zI - \mathcal{A})^{-1}(\mathcal{B}u_1(t-k) + \mathcal{D}\xi(t))$$

This expression motivates the definition of the *resolvent operator* for the total system as $\Phi = (zI - \mathcal{A}(t))^{-1}$.

3.1 Predictor Corrector Estimator Form

The standard *discrete-time Kalman filter* equations for a time-varying linear state-space model in predictor/corrector form, assuming the exogenous signals have known means, and with an error input (*Grimble and Johnson* 1988 [25]) are given as:

$$\hat{x}(t+1 \mid t) = A(t)\hat{x}(t \mid t) + B(t)u_1(t-k) + D(t)\bar{\xi}(t)$$

(Predictor) (35)

$$\hat{x}(t+1 \mid t+1) = \hat{x}(t+1 \mid t) + K_f(t+1)\left(e_0(t+1) - \hat{e}_0(t+1 \mid t)\right)$$

(Corrector) (36)

where $\hat{e}_0(t+1 \mid t) = C_e(t+1)\hat{x}(t+1 \mid t)$

$$-E(t+1)u_1(t+1-k) - \bar{v}(t) \quad (37)$$

The *Kalman filter* gain and *Ricatti* equations for a system with process and measurement noise covariance's Q and R:

$$K(t+1) = P(t+1 \mid t)C_e^T(t+1)$$
$$[C_e(t+1)P(t+1 \mid t)C_e^T(t+1) + R(t+1)]^{-1} \quad (38)$$

A priori covariance:

$$P(t+1 \mid t) = A(t)P(t \mid t)A^T(t) + D(t)Q(t)D^T(t) \quad (39)$$

A posteriori covariance: $P(t+1 \mid t+1) = P(t+1 \mid t)$

$$-K(t+1 \mid t)C_e(t+1)P(t+1 \mid t) \quad (40)$$

Initial conditions: $\hat{x}(0 \mid 0) = m_0$ and $P(0 \mid 0) = P_0$

Bias terms: $\bar{\xi}(t) = E\{\xi(t)\}$ and $\bar{v}(t) = E\{v(t)\}$.

Estimator Assumptions: It is possible to apply the above time-varying *Kalman filter* equations to the combined model for the state-dependent sub-systems in equations (31) and (32). The usual assumption will be made that the physical system to be controlled is observable and controllable from the process noise inputs.

The *state-dependent* model is of course a function of system state variables and these are time-varying with nonlinear operating point, however, the equations retain a linear state space type of structure.

4.0 Nonlinear Generalized Minimum Variance Control

The cost-minimisation problem may now be introduced, for the system which is shown in concise operator form in Fig. 2. The optimal *NGMV* control problem involves the minimisation of the variance of $\{\phi_0(t)\}$ in Fig. 2. The signal is to be minimized in a variance sense, where:

$$\phi_0(t) = (\mathcal{P}_c e)(t) + (\mathcal{Z}_c x)(t) + (\mathcal{F}_c u)(t) \qquad (41)$$

and the cost-index to be minimised (*Grimble*, 2005 [17]):

$$J = E\{\phi_0^T(t)\phi_0(t)\} = E\{trace\{\phi_0(t)\phi_0^T(t)\}\} \qquad (42)$$

where $E\{\cdot\}$ denotes the unconditional expectation.

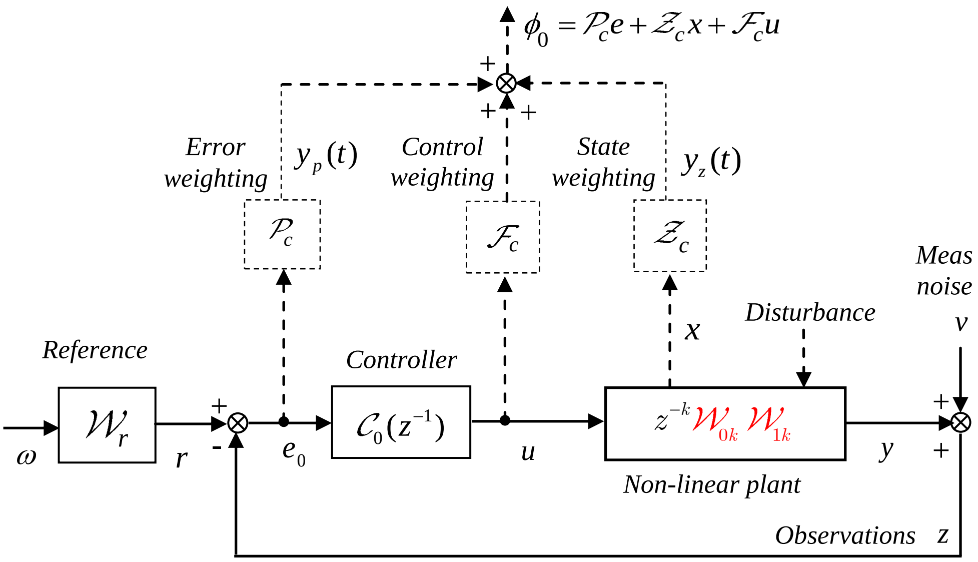

Fig. 2: **Single Degree of Freedom Closed-Loop Control System for the Nonlinear Plant**
(*Output signal to be minimised is shown dotted*)

The use of such a criterion in applications has been explored by *Alpbaz et. al.* (1998 [21]) and *Grimble* (2001 [10]). The so-called *inferred output signal* $\{\phi_0(t)\}$ that is minimised includes a *dynamic cost-function weighting* $\mathcal{P}_c(z^{-1})$, that acts on the error signal. This weighting is represented by a state-space sub-system, as described in §2, with weighted output $y_p(t) = \mathcal{P}_c e(t)$.

That is, if the system model is augmented with the weighting $\mathcal{P}_c(z^{-1})$ dynamics, then the first component of $\phi_0(t)$ can be represented by (20) as: $y_p(t) = \mathcal{C}_p x(t) + (\mathcal{E}_\phi u_1)(t-k)$, through the definition of appropriate output maps in (16). The signal $\{\phi_0(t)\}$ also includes the state weighting term: $y_z(t) = (\mathcal{Z}_c x)(t)$ that can be nonlinear, and it enables a cost-weighting to be introduced on all the state-dependent model subsystem states. If the state-dependent plant and reference/weighting subsystem states are denoted by $\mathcal{C}_{0z}$ and $\mathcal{C}_{1z}$, respectively, then the state weighting term:

$$y_z(t) = (\mathcal{Z}_c x)(t) = \mathcal{C}_{0z} x_0(t) + \mathcal{C}_{1z} x_1(t) = \mathcal{C}_z x(t) \qquad (43)$$

The combined *weighted error* and *state equation model* may be expressed as:

$$y_c(t) = y_p(t) + y_z(t) = \mathcal{P}_c e(t) + (\mathcal{Z}_c x)(t) \qquad (44)$$

In terms of the total state vector for the system:

$$y_c(t) = \mathcal{P}_c e(t) + (\mathcal{Z}_c x)(t) = \mathcal{C}_\phi x(t) + \mathcal{E}_\phi u_1(t-k) \qquad (45)$$

where

$$\mathcal{C}_\phi = \begin{bmatrix} \mathcal{C}_{\phi 0} & \mathcal{C}_{\phi 1} \end{bmatrix} = \begin{bmatrix} (\mathcal{C}_{0p} + \mathcal{C}_{0z}) & (\mathcal{C}_{1p} + \mathcal{C}_{1z}) \end{bmatrix} \qquad (46)$$

The final term in the criterion is the nonlinear dynamic control signal costing operator term: $(\mathcal{F}_c u)(t)$. If the smallest delay in each output channel of the plant is of magnitude k_0 steps this implies the control at time t affects the output at least k_0 steps later and the control signal costing is defined to have the form:

$$(\mathcal{F}_c u)(t) = z^{-k_0}(\mathcal{F}_{ck} u)(t) \qquad (47)$$

where $k_0 = k$ if the state models include through terms, or $k_0 = k+1$ if the through terms are null and one additional explicit step delay is therefore present in the plant model. The control costing $\mathcal{F}_{ck}$ will often be a linear operator but it may also be chosen to be nonlinear to cancel static input nonlinearities in appropriate cases. However, it may also be a dynamic nonlinear operator of quite a general form. The control weighting operator $\mathcal{F}_{ck}$ is assumed to be full rank, symmetric and invertible. The choice of dynamic weightings is critical to ensuring stability and the design, and the weighting $\mathcal{P}_c$ is typically a low-pass filter and $\mathcal{F}_c$ is a high-pass filter.

4.1 Solution of the NGMV Control Problem

The solution of the optimal control problem is straightforward and follows a *minimum variance* strategy working in the time-domain. It is obtained by introducing a prediction equation and by expanding the resulting expression for the signal that enters the cost-function. This signal may be referred to as a *minimised output* $\{\phi_0(t)\}$, since it is not a signal that exists physically. Recall that the signal,

$$\phi_0(t) = \mathcal{P}_c e(t) + (\mathcal{Z}_c x)(t) + (\mathcal{F}_c u)(t) = y_c(t) + (\mathcal{F}_c u)(t) \quad (48)$$

where from (45) the *minimised output*, may be written as:

$$\phi_0(t) = \mathcal{C}_\phi x(t) + \mathcal{E}_\phi u_1(t-k) + (\mathcal{F}_c u)(t) \quad (49)$$

Recall that the signal: $u_1(t) = \mathcal{W}_{1k}' u(t)$ and that the control signal weighting was defined as:

$(\mathcal{F}_c u)(t) = z^{-k_0}(\mathcal{F}_{ck}\, u)(t)$, so that (49) becomes:

$$\phi_0(t) = \mathcal{C}_\phi x(t) + ((\mathcal{E}_\phi \mathcal{W}_{1k}' + z^{-k_0+k}\mathcal{F}_{ck})u)(t-k) \quad (50)$$

4.2 The Prediction Equations

The prediction equation was obtained using equation (28) with the finite impulse response term, as:

$$\hat{x}(t+k_0 \mid t) = \mathcal{A}^{k_0}\hat{x}(t \mid t) + \mathcal{T}(k_0, z^{-1})u_1(t) \quad (51)$$

where from equations (21) and (27):

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_0 & 0 \\ \mathcal{A}_{21} & \mathcal{A}_1 \end{bmatrix} \quad \text{and}$$

$$\mathcal{A}^{k_0} = \coprod_{j=0}^{k_0-1} \mathcal{A}(t+j) = \mathcal{A}(t+k_0-1)\mathcal{A}(t+k_0-2)\ldots\mathcal{A}(t) \quad (52)$$

The predicted values of the state related terms in equation (50) may therefore be written as:

$$\mathcal{C}_\phi(t+k_0)\,\hat{x}(t+k_0 \mid t) = \mathcal{C}_\phi(t+k_0)\mathcal{A}^{k_0}\hat{x}(t \mid t)$$
$$+\mathcal{C}_\phi(t+k_0)\mathcal{T}(k_0, z^{-1})u_1(t) \quad (53)$$

where the operator $\mathcal{T}(k_0, z^{-1})$ was defined by (26). The k steps-ahead prediction of the signals: $y_c(t)$ and $\phi_0(t)$ follow from (44), (50) and (53), as follows:

$$\hat{y}_c(t+k_0 \mid t) = \mathcal{C}_\phi(t+k_0)\hat{x}(t+k_0 \mid t)$$
$$+(\mathcal{E}_\phi(t+k_0)\mathcal{W}_{1k}' u)(t+k_0-k)$$
$$= \mathcal{C}_\phi(t+k_0)\hat{x}(t+k_0 \mid t) + \mathcal{E}_\phi(t+k_0)\mathcal{W}_{1k}' z^{k_0-k}u(t)$$

Note that when the through term in the plant model is null there is effectively one more delay element and $k_0 = k+1$, and from (45) in this case $\mathcal{E}_\phi(t+k_0)$ will also be null. Thus, the term $\mathcal{E}_\phi(t+k_0)\mathcal{W}_{1k}' z^{k_0-k}$ can be replaced as: $\mathcal{E}_\phi(t+k_0)\mathcal{W}_{1k}'$ giving:

$$\hat{y}_c(t+k_0 \mid t) = \mathcal{C}_\phi(t+k_0)\hat{x}(t+k_0 \mid t) + \mathcal{E}_\phi(t+k_0)\mathcal{W}_{1k}' u(t)$$

$$= \mathcal{C}_\phi(t+k_0)\mathcal{A}^{k_0}\hat{x}(t \mid t)$$
$$+((\mathcal{C}_\phi(t+k_0)\mathcal{T}(k_0, z^{-1}) + \mathcal{E}_\phi(t+k_0))\mathcal{W}_{1k}')u(t) \quad (54)$$

and $\hat{\phi}_0(t+k_0 \mid t) = \hat{y}_c(t+k_0 \mid t) + (\mathcal{F}_{ck}u)(t)$

$$= \mathcal{C}_\phi(t+k_0)\mathcal{A}^{k_0}\hat{x}(t \mid t) + ((\mathcal{C}_\phi(t+k_0)\mathcal{T}(k_0, z^{-1})$$
$$+\mathcal{E}_\phi(t+k_0))\mathcal{W}_{1k}' + \mathcal{F}_{ck})u(t) \quad (55)$$

4.3 NGMV Control Problem Cost-Function

The cost-function involves the minimization of the variance of the *weighted error, states* and *control signals*. The variance: $J = E\{\phi_0(t+k_0)^T\phi_0(t+k_0)\}$ (56)

may be written in terms of the prediction $\hat{\phi}_0(t+k_0 \mid t)$ and the prediction error: $\tilde{\phi}_0(t+k_0 \mid t)$, noting these signals are orthogonal in a statistical sense:

$$J = E\{\hat{\phi}_0(t+k_0 \mid t)^T\hat{\phi}_0(t+k_0 \mid t)\} + E\{\tilde{\phi}_0(t+k_0 \mid t)^T\tilde{\phi}_0(t+k_0 \mid t)\} \quad (57)$$

The prediction error $\tilde{\phi}_0(t+k_0 \mid t)$ does not depend upon control action and hence the cost is clearly minimized by setting the predicted values of the signal $\phi_0(t)$, for k_0 steps-ahead, to zero. Setting the predicted values of (55) to zero provides two possible expressions for the control.

Reminder of Assumptions: Observe that this latter expression includes the *Kalman filter* state estimate and the *finite pulse response* block, which depend upon the state-dependent models. The NL input plant subsystem $\mathcal{W}_{1k}$ was assumed to be finite gain stable but no structure was assumed. However, the state-dependent nonlinear system $\mathcal{W}_{0k}$ may be unstable. For optimal state estimation the assumption was made that the required future values of the *state-dependent* models could be computed. Finally note that the control cost-weighting was defined, so that the inverse of $\mathcal{F}_{ck}$ exists, and thence the inverse of the operators in (59) or (60) may be found.

Theorem 1: NGMV State-Dependent Controller

Let the operator $\mathcal{N}_0$ represent the mapping from the signal $u_1(t)$ to the signal $\phi_0(t)$ to be minimised:

$$(\mathcal{N}_0 u_1)(t) = ((\mathcal{P}_c\mathcal{W}_{0k}'(t+k_0) - \mathcal{Z}_c\tilde{\mathcal{W}}_{0k}'(t+k_0))u_1)(t) \quad (58)$$

Assume that the weighting operators $\mathcal{P}_c, \mathcal{Z}_c$ and $\mathcal{F}_c$ are chosen so that the *NL* operator: $(\mathcal{N}_0\mathcal{W}_{1k}' - \mathcal{F}_{ck})$ has a finite-gain m_2 stable causal inverse, to ensure the system is closed-loop stable. The *NGMV* optimal controller to minimize the variance of the weighted error, states and control signals may then be computed. The *NGMV optimal control signal* may be expressed in the two alternative forms (59) and (60):

$$u(t) = \left(\mathcal{F}_{ck} + (\mathcal{C}_\phi(t+k_0)\mathcal{T}(k_0, z^{-1}) + \mathcal{E}_\phi(t+k_0))\mathcal{W}_{1k}'\right)^{-1}$$
$$\times(-\mathcal{C}_\phi(t+k_0)\mathcal{A}^{k_0}\hat{x}(t \mid t)) \quad (59)$$

or

$$u(t) = -\mathcal{F}_{ck}^{-1}\left[\mathcal{C}_\phi(t+k_0)\mathcal{A}^{k_0}\hat{x}(t \mid t)\right.$$

$$+(\mathcal{C}_\phi(t+k_0)\mathcal{T}(k_0,z^{-1})+\mathcal{E}_\phi(t+k_0))\mathcal{W}_{1k}u(t)\big] \qquad (60)$$

where $\mathcal{C}_{0p} = -\mathcal{E}_p\mathcal{C}_0$ and $\mathcal{C}_{1p} = \left[\mathcal{E}_p\mathcal{C}_r \quad \mathcal{C}_p\right]$,

$\mathcal{C}_\phi = \left[(\mathcal{C}_{0p}+\mathcal{C}_{0z}) \quad (\mathcal{C}_{1p}+\mathcal{C}_{1z})\right]$ and $\mathcal{E}_\phi = -\mathcal{E}_p\mathcal{E}_0$. •

Proof: The proof of optimality involves collecting the above results, subject to the assumptions. To minimize (56) note the simplification to (57) and that this is a minimum when $\hat{\phi}_0(t+k_0\,|\,t)=0$. This signal is given by (55) and the expression for optimality involves setting (55) to zero. This equation can be satisfied by either of the two expressions for the optimal control (59) or (60). The stability of the closed loop system is discussed below.

Remarks: The expressions for the *NGMV* optimal control signal lead to the two alternative structures, but that shown in Fig. 3 is the most useful for implementation. It is important to note that although the controller does seem to depend upon a nonlinear system inverse the existence of this inverse can be demonstrated by construction. That is the main inverse involved is that due to the inner block in Fig. 3 and this can be implemented as shown. There is a practical issue of computing the solution of the algebraic loop involved, which can be faster than is available in Simulink if the particular form of the nonlinearity is taken into account [7].

The order of the *Kalman filter* depends only on the delay-free subsystems. The channel delays do not therefore inflate the order of the filter required. The system matrices involved in the state dependent model can be treated as known. They involve future values of the control and states, which can be predicted.

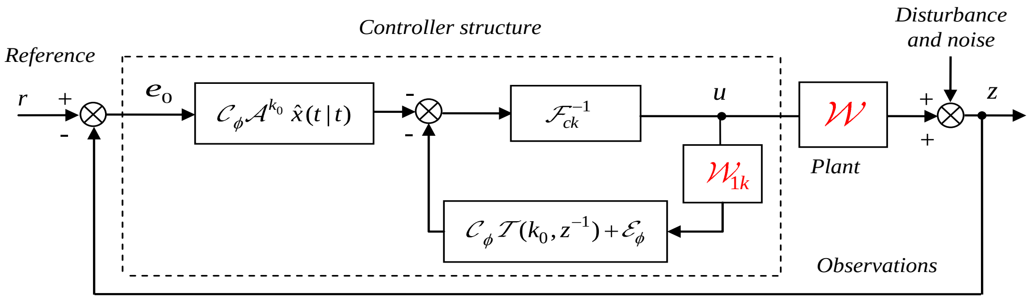

Fig. 3: **Control Signal Generation and Controller Modules**

Implementation Issues and Stability: The controller involves a *time-varying Kalman filter* with predictor stage and this may be illustrated in the more physically intuitive structure of the controller in Fig. 4. Introducing the *Kalman filter* structure leads to this realisation of the controller which is easy to implement and the state estimates may also be useful for condition monitoring purposes. A simple way to initialise the filter is to run the simulation and copy the steady state values of states.

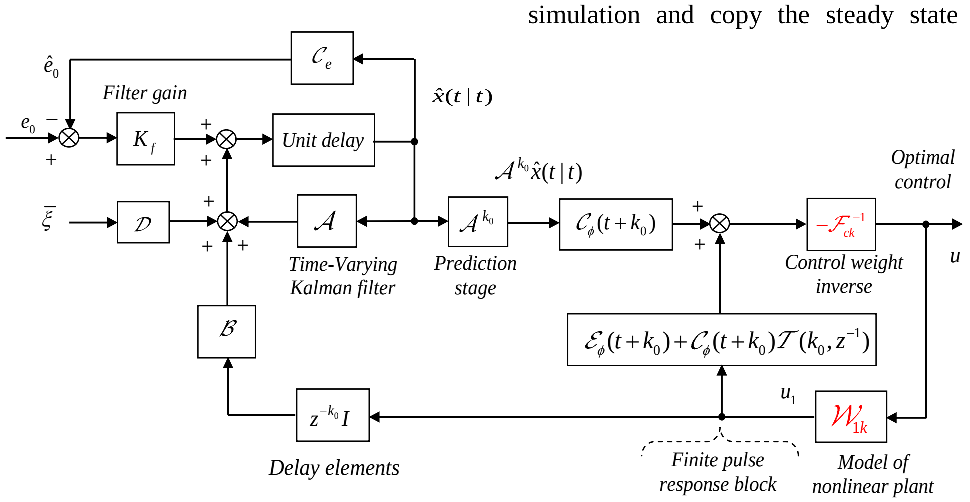

Fig. 4: **NGMV Optimal Controller in State-Space Kalman Filtering Form**

4.4 *Systems Analysis*

For linear systems stability is ensured with *GMV* designs when the combination of a control weighting and an error weighted plant model is strictly minimum-phase (*Grimble* 2005 [18]). For nonlinear systems a related operator equation for the *t-varying* system must have a stable inverse. These stability results are established below. First write the state estimate in the operator form:

$$\hat{x}(t\,|\,t) = T_{f1}(z^{-1})e_0(t) + T_{f2}(z^{-1})u_1(t-k) \qquad (61)$$

To identify the form of these operators and for the stability analysis neglect the input bias or *DC* signals and all stochastic inputs other than the reference signal. Note from (36) and (37) for the system model of interest:

$\hat{x}(t|t) = (I - K_f(t)\mathcal{C}_e(t))\hat{x}(t|t-1)$

$+K_f(t)\left(e_0(t) + \mathcal{E}(t)u_1(t-k)\right)$ but from (35),

$\hat{x}(t|t-1) = \mathcal{A}(t-1)\hat{x}(t-1|t-1) + \mathcal{B}(t-1)u_1(t-k-1)$

Thence

$$\hat{x}(t|t) = \left(I - (I - K_f(t)\mathcal{C}_e(t))z^{-1}\mathcal{A}(t)\right)^{-1}\left[K_f(t)e_0(t) + \left((I - K_f(t)\mathcal{C}_e(t))\mathcal{B}(t-1)z^{-1} + K_f(t)\mathcal{E}(t)\right)u_1(t-k)\right] \quad (62)$$

It follows that the above operators in (61) have the form:

$$T_{f1}(z^{-1}) = \left(I - (I - K_f(t)\mathcal{C}_e(t))z^{-1}\mathcal{A}(t)\right)^{-1}K_f(t) \quad (63)$$

$$T_{f2}(z^{-1}) = \left(I - (I - K_f(t)\mathcal{C}_e(t))z^{-1}\mathcal{A}(t)\right)^{-1} \times\left((I - K_f(t)\mathcal{C}_e(t))\mathcal{B}(t-1)z^{-1} + K_f(t)\mathcal{E}(t)\right) \quad (64)$$

The following expression, obtained by simple manipulation [10], is also required:

$$T_{f2}(z^{-1}) - T_{f1}(z^{-1})\mathcal{W}_{0k} = \Phi\mathcal{B}(t) \quad (65)$$

4.5 Output Minimised

For the stability analysis an expression is required for the control action in terms of the closed-loop operators with input reference signal. Recall from equation(55) and substituting from (61) $T_{f2}(z^{-1}) = \Phi\mathcal{B}(t) + T_{f1}(z^{-1})\mathcal{W}_{0k}$,

$$\hat{\phi}_0(t+k_0|t) = \mathcal{C}_\phi\mathcal{A}^{k_0}T_{f1}r(t) + \left[\mathcal{C}_\phi(\mathcal{T}(k_0, z^{-1}) + \mathcal{A}^{k_0}\Phi\mathcal{B}(t)z^{-k})\mathcal{W}_{1k}' + \mathcal{E}_\phi\mathcal{W}_{1k}' + \mathcal{F}_{ck}\right]u(t) \quad (66)$$

Operator Relationship: The second operator relationship, again obtained by straightforward manipulation [10] is:

$$\Phi(t+k_0)\mathcal{B}(t+k_0)z^{k_0-k} = \mathcal{T}(k_0, z^{-1}) + \mathcal{A}^{k_0}\Phi(t)\mathcal{B}(t)z^{-k} \quad (67)$$

Using this result the *minimised output* (66) may be written

$$\hat{\phi}_0(t+k_0|t) = \mathcal{C}_\phi\mathcal{A}^{k_0}T_{f1}r(t) + \left[(\mathcal{C}_\phi(t+k_0)\Phi(t+k_0)\mathcal{B}(t+k_0)z^{k_0-k} + \mathcal{E}_\phi(t+k_0))\mathcal{W}_{1k}' + \mathcal{F}_{ck}\right]u(t) \quad (68)$$

Now from equation (45):

$y_c(t+k_0) = \mathcal{P}_c r(t+k_0) - \mathcal{P}_c y(t+k_0) + (\mathcal{Z}_c x)(t+k_0)$ and

$$(\mathcal{Z}_c x)(t+k_0) - \mathcal{P}_c y(t+k_0) = \left(\mathcal{Z}_c\tilde{\mathcal{W}}_{0k}(t+k_0) - \mathcal{P}_c\mathcal{W}_{0k}(t+k_0)\right)\mathcal{W}_{1k}'u(t)$$

where $\tilde{\mathcal{W}}_{0k}$ denotes the transfer between $u_1(t-k)$ and the states of the state dependent sub-system. Clearly,

$$\mathcal{C}_\phi(t+k_0)\Phi(t+k_0)\mathcal{B}(t+k_0)z^{k_0-k} + \mathcal{E}_\phi(t+k_0) = \mathcal{Z}_c\tilde{\mathcal{W}}_{0k}(t+k_0) - \mathcal{P}_c\mathcal{W}_{0k}(t+k_0) \quad (69)$$

The first term on the right of this expression represents the cost-function state weighting and the second the output weighting term. Recall the term $\mathcal{E}_\phi(t+k_0)\mathcal{W}_{1k}'z^{k_0-k}$ is null when there is no through term and $k_0 = k+1$. Thence, the cost which predicts forward k_0 steps ensures that the term $\mathcal{C}_\phi(t+k_0)\Phi(t+k_0)\mathcal{B}(t+k_0)z^{k_0-k}$ includes a through term, which is important for the definition of the optimal cost. It is therefore helpful to define the transfer-operator:

$$\mathcal{N}_0 = (\mathcal{P}_c\mathcal{W}_{0k}(t+k_0) - \mathcal{Z}_c\tilde{\mathcal{W}}_{0k}(t+k_0)) \quad (70)$$

This represents the transfer between the signal u_1 and the output to be minimised $\{\phi_0(t)\}$.

Then equation (68) becomes:

$$\hat{\phi}_0(t+k_0|t) = \mathcal{C}_\phi\mathcal{A}^{k_0}T_{f1}(z^{-1})r(t) + (-\mathcal{N}_2\mathcal{W}_{1k}' + \mathcal{F}_{ck})u(t) \quad (71)$$

4.6 Stability Analysis

To simplify the stability analysis, recall that the external inputs, except the reference *r*(t), were assumed null in the previous section. Using (71) the condition for optimality $\hat{\phi}_0(t+k|t) = 0$ leads to the optimal control as:

$$u(t) = \mathcal{F}_{ck}^{-1}\left(-\mathcal{C}_\phi\mathcal{A}^{k_0}T_{f1}(z^{-1})r(t) + (\mathcal{N}_2\mathcal{W}_{1k}'u)(t)\right) \quad (72)$$

Rearranging, the *optimal control* and *plant output* signals, in terms of the exogenous input, become:

$$u(t) = \left(\mathcal{N}_2\mathcal{W}_{1k}' - \mathcal{F}_{ck}\right)^{-1}\mathcal{C}_\phi\mathcal{A}^{k_0}T_{f1}(z^{-1})r(t) \quad (73)$$

$$(\mathcal{W}u)(t) = \mathcal{W}(\mathcal{N}_2\mathcal{W}_{1k}' - \mathcal{F}_{ck})^{-1}\mathcal{C}_\phi\mathcal{A}^{k_0}T_{f1}(z^{-1})r(t) \quad (74)$$

where the delay-free plant model $\mathcal{W}_k' = \mathcal{W}_{0k}\mathcal{W}_{1k}'$ and $\tilde{\mathcal{W}}_{0k}$ denotes the transfer-operator between $u_1(t-k)$ and the states of the *state-dependent* sub-system.

To show that the closed-loop system is stable, recall that various plant operator blocks are assumed to be *finite gain stable*. Note the series connection of two finite gain m_2 stable systems is m_2 stable. The assumption was that the cost-weightings are chosen, so that the operator $(\mathcal{N}_2\mathcal{W}_{1k}' - \mathcal{F}_{ck})^{-1}$ is *finite gain stable*. The cost will often involve just the weighted error and control signals, since the state weighting will be omitted. In this case the following operator must be *finite gain stable*:

$$\left((\mathcal{P}_c\mathcal{W}_{0k}(t+k_0)\mathcal{W}_{1k}' - \mathcal{F}_{ck}\right)^{-1} = \left((\mathcal{P}_c\mathcal{W}_k'(t+k_0) - \mathcal{F}_{ck}\right)^{-1} \quad (75)$$

Also observe that the internal feedback loop in the controller in Fig. 3 does not contain any subsystem that is unstable. It only contains $\mathcal{W}_{1k}'$ which was assumed finite gain stable. The consequence is that the inverse dynamics will not attempt to cancel unstable parts of the plant model.

The operator in (75) must of course be "finite gain" stable for the above result to ensure closed loop stability of the system. To provide an easy method of finding starting values of cost-weightings it can be shown that if there exists say a *PID* controller that will stabilize the *NL* system, without transport-delay elements, then a set of cost weightings can be defined to guarantee the existence of this inverse and ensure the stability of the closed-loop.

Note that the controller includes a model of the nonlinear system and does not involve linearization (*Shamma* and *Athans*, 1992 [26]). Moreover, it is not like the original state-dependent control solutions where the system was assumed frozen at a current operating point so that the steady-state Riccati equation solution could be used [14]. Proving stability in either case is problematic. It is of course not simple to guarantee stability for state dependent or *LPV* systems but at least the controller is like a fixed model based solution so its behaviour should be more predictable and the results referred to above provide a starting point for cost weighting selection.

The controller can be represented in an alternative form that is good for implementation and useful for the analysis in the next section. First note that the controller structure shown in Fig. 3 involves the predicted states from the *Kalman Filter*, but the two different paths from the observations and control inputs can be separated in Fig. 5.

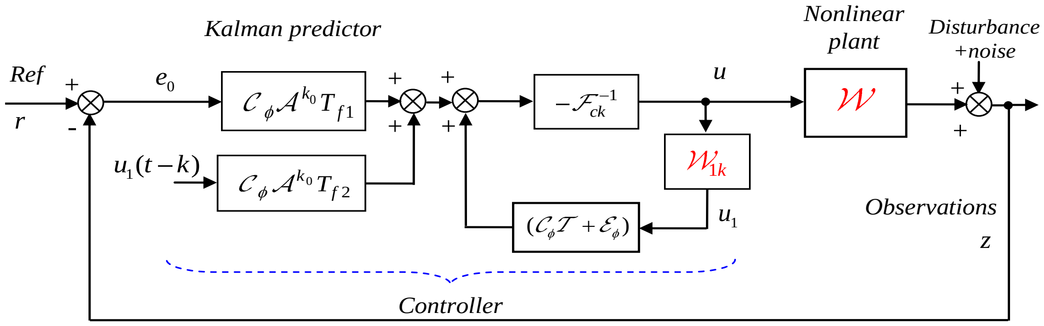

Fig. 5: **Feedback Control Signal Generation and Controller Modules**

5.0 Design Example

The application of the *NGMV* controller to the control of a simple chemical process is considered. The process is an irreversible exothermic first order reaction, which takes place in a continuous stirred tank reactor (*CSTR*), as shown in Fig. 6. It is assumed that the liquid in the reactor is perfectly mixed and the feed flow is equal to the product outflow. The cooling jacket temperature T_c is regarded as an input to the process and the product concentration C_a is regarded as the output. Similar types of process have been studied extensively in the literature as they present very challenging control problems.

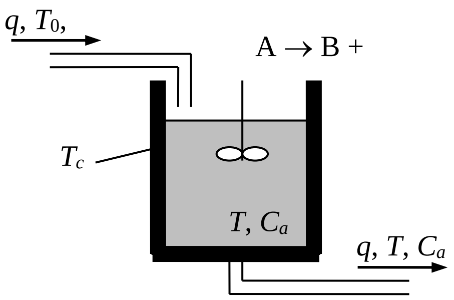

Fig. 6: **Continuous Stirred Tank Reactor Possibly Unstable Process** (input variable is cooling jacket temperature T_c and output is the concentration C_a)

The *CSTR* processes exhibit rich nonlinear behaviour, involving multiple steady-state solutions and both stable and unstable equilibrium points. The nonlinear dynamic responses vary widely with operating point. Consider a normalized dimensionless model given in Hernandez and Arkun (1993, [4]), where the values of the parameters are as listed in Table 1:

$$\begin{aligned}
\dot{x}_1 &= -x_1 + D_a(1-x_1)\exp\left(x_2/(1+x_2/\varphi)\right) \\
\dot{x}_2 &= -x_2 + BD_a(1-x_1)\exp\left(x_2/(1+x_2/\varphi)\right) + \beta(u-x_2) \\
y &= x_1
\end{aligned} \qquad (76)$$

By defining a new scaled input: $u_s = (u+4)/8$, both the input and output signals are contained in range (0,1).

Table 1: CSTR model parameters

Parameter	Meaning	Value
D_a	Damköhler number	0.072
φ	dimensionless activation energy	20.0
B	heat of reaction coefficient	8.0
β	heat transfer coefficient	0.3

A polynomial *ARMA* model was identified from simulation data in [21] and this was followed by the use of a sigmoid function. Such a structure with output nonlinearity is not suitable for the state-dependent *NGMV* theory. A new nonlinear *ARMAX* model using the tools provided with the *System Identification Toolbox* for *Matlab* was therefore identified. The dimensionless sample time of 0.5 units was used to generate the data for identification. An interesting feature of the *CSTR* model is the presence of two stable regions, at the two ends of the output range, separated by an unstable region. It is difficult to tune the system in this unstable region, from which the system output appears repelled. The effort was made to include the whole operating range in the estimation data. The following model was found to provide a good balance between accuracy and complexity:

$$\begin{aligned}
y(t) &= \theta_0 + \theta_1 y(t-1) + \theta_2 y(t-2) + \theta_3 u_s(t-1) + \theta_4 u_s(t-2) \\
&+ \theta_5 y^2(t-1) + \theta_6 y(t-1)y(t-2) + \theta_7 y(t-1)u_s(t-1) \\
&+ \theta_8 y(t-1)u_s(t-2) + \theta_9 y^3(t-1) + \theta_{10} y^2(t-1)y(t-2)
\end{aligned} \qquad (77)$$

with the estimated parameter values:

$\theta_0 = 0.0129$, $\theta_1 = -0.0390$, $\theta_2 = 0.5112$, $\theta_3 = 0.0219$,
$\theta_4 = 0.0290$, $\theta_5 = 7.8027$, $\theta_6 = -6.5853$, $\theta_7 = 0.1232$,
$\theta_8 = 0.0877$, $\theta_9 = -7.9623$, $\theta_{10} = 7.0073$

The model verification is shown in Fig. 7 and a good match is confirmed between the plant and model responses for both the estimation and the validation data.

A comparison between the continuous-time system static characteristics is shown in Fig. 8. The curves were plotted by considering a range of steady-state outputs, and computing the corresponding inputs from equations (76) and (77). Multiple equilibrium points are evident from the plots and the unstable equilibriums on the model curve have been highlighted in green. Note that these correspond with the local negative gain (slope) of the model, which may be an issue when selecting the control weighting sign.

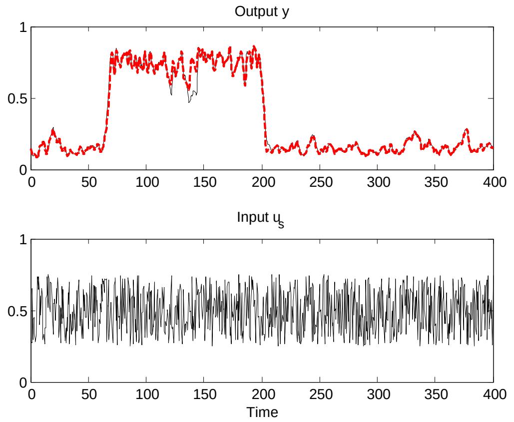

Fig. 7: **NARX Model Validation**: plant (solid), model (dashed). (The estimation data used for identification: up to time 200)

For the purpose of the *NGMV* control, the above polynomial *NARMAX* model was converted to a state-dependent model by defining the following states:

$$x_1(t) = y(t), \quad x_2(t) = y(t-1), \quad x_3(t) = u_s(t-1).$$

The state-dependent model then follows as:

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \\ x_4(t+1) \end{bmatrix} = \begin{bmatrix} x_1^2+x_1+1 & x_1^2+x_1+1 & x_1+1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} x_1+1 \\ 0 \\ 1 \\ 0 \end{bmatrix} u_s(t) \quad (78)$$

$$y(t) = x_1(t) \quad (79)$$

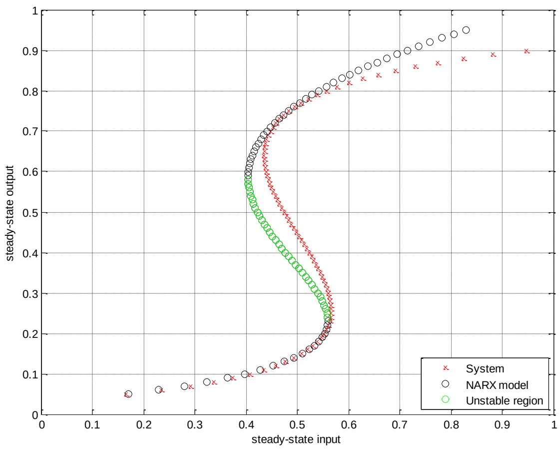

Fig. 8: **Static Characteristics for the System**

5.1 *Control design and simulation results*

The major challenge is to control the system around an unstable equilibrium point, i.e. for the middle concentration values. As the nominal "unstable" operating point choose the point corresponding to the control input of $u_s(t) = 0.5$. Based on the model equation, the corresponding output steady-state values can be computed as $y_{01} = 0.1437$, $y_{02} = 0.3652$ and $y_{03} = 0.7658$. Out of these 3 solutions, y_{02} is the unstable equilibrium and the control objective will consist of regulating the system around this value. The *NGMV* controller was designed for this problem using:

Reference: $W_r(z^{-1}) = 0.05/(1-0.95z^{-1})$

Error weighting: $P_c(z^{-1}) = (1-0.1z^{-1})^2/(1-0.97z^{-1})^2$

Control weight: $F_{ck}(z^{-1}) = 0.55 \cdot (1-0.3z^{-1})^2/(1-0.9z^{-1})^2$

The reference model is normally chosen to be a near integrator since most reference signals are slowly changing. The error weighting is also chosen as a near integrator and this introduces integral action type behaviour in the controller. Finally, the control signal costing is chosen to include a lead term to ensure the controller rolls off but can also include a lag term to introduce derivative type action for augmenting stability in the mid frequency range.

The goal was to move the process from the stable equilibrium $y_{01} = 0.1437$ to the unstable operating point: $y_{02} = 0.3652$, and keep it there in the presence of a step output disturbance. The results are shown in Fig. 9 and despite a rather oscillatory control signal response to the disturbance the objectives were satisfied.

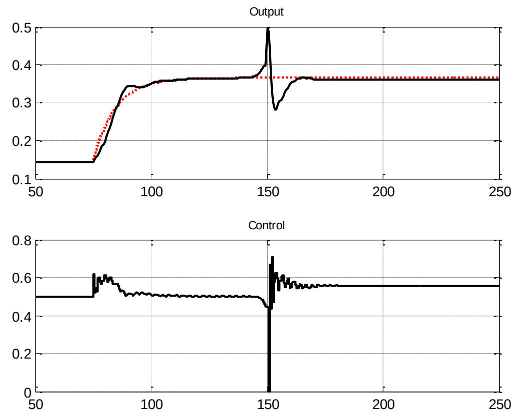

Fig. 9: **NGMV Output Tracking and Regulation About an Unstable Operating Point**
(Upper plot: output (solid), set-point (dotted))

The final test consisted of tracking the concentration set-point across the operating range. Since the dynamics and nonlinearity of the system vary wildly for different operating conditions some form of nonlinear control costing is needed. A simple approach is to schedule the changes in weighting by defining a separate set of the control weighting parameters for each operating regime (four of them were specified), and to then use a simple switching scheme for these parameters. The control weighting was parameterized as:

$$F_{ck}(z^{-1}) = \rho \cdot \left((1-\beta z^{-1})^2 / (1-\alpha z^{-1})^2 \right)$$

and the parameters ρ, β and α were tuned separately for each region corresponding to a single step change (the switching variable used was the output concentration). The Table 2 contains the values of the parameters and the corresponding frequency responses of the control weighting are as shown in Fig. 10.

Table 2: Control Weighting Parameters

Region centre	β	α	ρ
0.1	-0.3	0.92	0.4
0.25	-0.3	0.92	0.5
0.4	-0.3	0.92	0.65
0.55	-0.3	0.92	0.3
0.8	-0.5	0.9	1.2

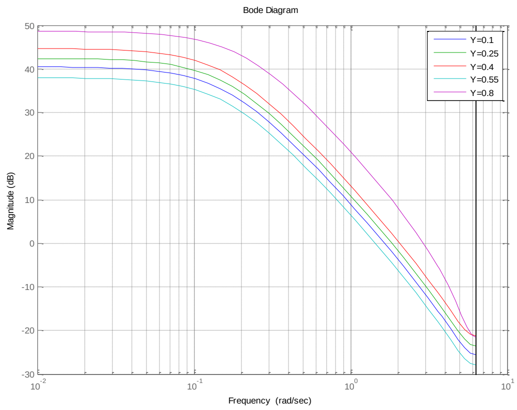

Fig. 10: **Bode Plots of Gain-Scheduled Control Weights**

The above control costing values were collected in look-up tables and switched using gain-scheduling. Note that this is not quite the same as using a traditional gain scheduling type of *linear* control law. The nominal simulation results (with the *NARX* model as the "plant") are shown in Fig. 11 and good tracking is achieved across the operating range.

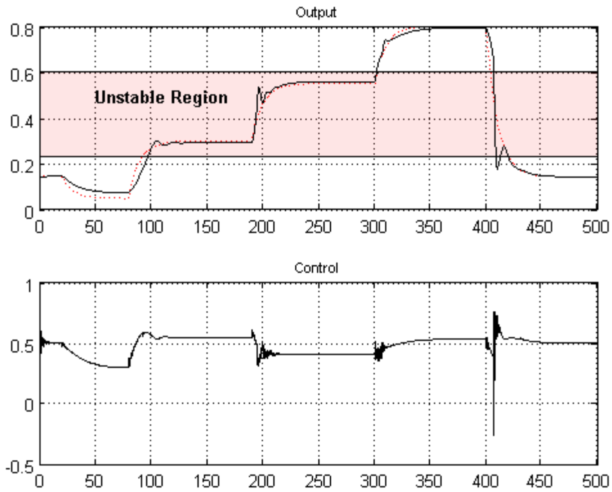

Fig. 11: **NGMV Control of Output Tracking Across the Operating Range**
(with a gain-scheduled control weighting and nominal model)

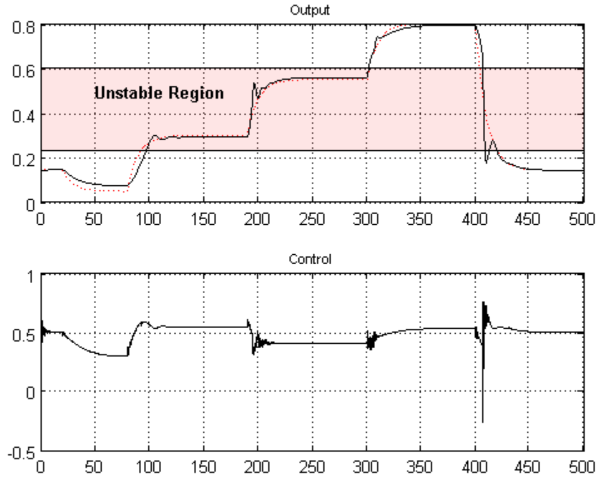

Fig. 12: **NGMV Control Output Tracking Across the Operating Range** (with a gain-scheduled control weighting for the original model)

On the other hand, the simulations for the original system, shown in Fig. 12, result in a deterioration in performance within the unstable region (although tracking in the "stable" region is actually improved). This is an expected effect of plant-model mismatch and can be corrected by identifying a better model – by for example including more regresses terms, not necessarily polynomial ones.

6.0 Links to the Smith Predictor

The optimal controller can be expressed in a similar form to that of a *Smith Predictor* to provide some confidence in the design. This enables a nonlinear state-space version of the *Smith Predictor* to be introduced, but this structure also limits the applications to *open-loop stable* systems. A further result is required based on (67) $\mathcal{T}(k_0,z^{-1}) = (\Phi(t+k_0)z^{k_0-k} - \mathcal{A}^{k_0}\Phi(t)z^{-k})\mathcal{B}(t+k)$ and on (65) $T_{f2}(z^{-1}) = \Phi(t)\mathcal{B}(t) + T_{f1}(z^{-1})\mathcal{W}_{0k}$. That is,

$$(\mathcal{C}_\phi\mathcal{T}(k_0,z^{-1}) + \mathcal{E}_\phi - \mathcal{C}_\phi\mathcal{A}^{k_0}T_{f1}\mathcal{W}_{0k}\,z^{-k} + \mathcal{C}_\phi\mathcal{A}^{k_0}T_{f2}\,z^{-k})$$
$$= \mathcal{C}_\phi(t+k_0)\Phi(t+k_0)z^{k_0-k}\mathcal{B}(t+k) + \mathcal{E}_\phi(t+k_0) \qquad (80)$$

The state dependent subsystems in the controller in Fig. 5, may be modified by adding and subtracting terms, and combining inner-loop blocks, using (80):

$$\mathcal{C}_\phi\mathcal{T}(k_0,z^{-1}) + \mathcal{E}_\phi - \mathcal{C}_\phi\mathcal{A}^{k_0}T_{f1}\mathcal{W}_{0k}\,z^{-k} + \mathcal{C}_\phi\mathcal{A}^{k_0}T_{f2}\,z^{-k}$$
$$= \mathcal{C}_\phi(t+k_0)\Phi(t+k_0)\mathcal{B}(t+k_0)z^{k_0-k} + \mathcal{E}_\phi(t+k_0)$$

This result suggests the controller can be simplified to the *Smith predictor* type structure in Fig. 13. Note the control signal u to the feedback signal p that the transfer is null when the model $z^{-k}\mathcal{W}_k = z^{-k}\mathcal{W}_{0k}\mathcal{W}_{1k}$ matches the plant model. It follows that the control action, due to reference signal r changes, is not due to feedback but involves the open-loop stable compensator involving the block $\mathcal{C}_\phi\mathcal{A}^kT_{f_1}$ and the *inner* nonlinear feedback loop. It follows that the control action due to the reference changes will be due to the cascade of these two blocks.

Normally the state weighting is null and the inner-loop feedback path transfer may then be written as:

$$\mathcal{C}_\phi(t+k_0)\Phi(t+k_0)\mathcal{B}(t+k_0)z^{k_0-k} + \mathcal{E}_\phi(t+k_0)$$

$$= \mathcal{Z}_c\tilde{\mathcal{W}}_{0k}(t+k_0) - \mathcal{P}_c\mathcal{W}_{0k}(t+k_0) \rightarrow -\mathcal{P}_c\,\mathcal{W}_{0k}(t+k_0) \qquad (81)$$

In this case the open-loop transfer operator for the inner-loop, includes the weightings: $\mathcal{F}_{ck}^{-1}\mathcal{P}_c$ acting like an inner-loop controller. These weightings can be chosen to be of a filtered *PID* controller form. The reverse of this argument suggests that cost weightings can be found that stabilise the closed loop given an initial stabilising control exists for the delay free system.

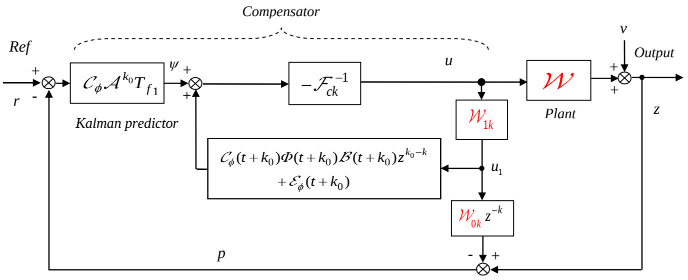

Fig. 13: **Nonlinear Smith Predictor Compensator and Internal Model Structure**

7.0 Concluding Remarks

A simple controller for nonlinear multivariable systems was introduced that extends the family of *NGMV* controllers to much more general systems. The major development over the basic *NGMV* control law in [6] involves the introduction of a more general model structure, where the nonlinearities may be associated with either inputs or outputs and include open-loop unstable elements. The solution is for systems represented in a combination of nonlinear operator, state dependent or linear state-space system forms. Previous work was restricted to plants where the nonlinear subsystem was only at the input and was assumed stable.

The inclusion of the state-dependent sub-system model has the advantage that it may be used to represent open-loop unstable plants with both input and output nonlinearities. The model state dependent model structure actually absorbs *LPV* type models and even certain classes of hybrid models. The class of controllers are not of course potentially so valuable as nonlinear predictive controllers that in general have better stability and performance characteristics. However, the basic concepts and controller structure are simple and they therefore provide a possible replacement for *PID* controls, which is a very different target problem. Adaptive control using the philosophy will be the subject of future research (*Zhu et. al.* [23] 1999).

Acknowledgements: We are grateful for the support of the EPSRC and the help of Dr Yan Pang of the ICC.

8.0 References

1. Åström, K J., 1979, Introduction to Stochastic Control Theory, Academic Press, London.

2. Hastings-James, R., 1970, *A Linear Stochastic Controller for Regulation of Systems with Pure Time Delay*, University of Cambridge, Eng. Report Nº CN/70/3.

3. Clarke, D. W., and Hastings-James, R, 1971, *Design of Digital Controllers for Randomly Disturbed Systems*, Proc. IEE, Vol. 118, Nº. 10, 1502-1506.

4. Hernandez, E. and Arkun,Y, 1993, *Control of Nonlinear Systems Using Polynomial ARMA Models*, AIChE 39, p.446-460.

5. Grimble, M J, 2004, *GMV Control of Nonlinear Multivariable Systems*, *Control 2004 Conference*, Univ of Bath.

6. Grimble, M J, 2005, Non-linear Generalised Minimum Variance Feedback, Feedforward and Tracking Control, Automatica, Vol. 41, pp 957-969.

7. Grimble, M J, and P Majecki, 2005, *Nonlinear Generalised Minimum Variance Control Under Actuator Saturation*, IFAC World Congress, Prague, 8 July, 2005.

8. Grimble, M J, and P Majecki, 2006, *H∞ Control of Nonlinear Systems with Common Multi-channel Delays*, American Control Conference, Minneapolis, June 14-16.

9. Grimble M. J., 2007, *GMV Control of Non-linear Continuous-time systems Including Common Delays and State-space Models*, International Journal of Control, Vol. 80, No. 1, pp. 150–165.

10. Grimble, M. J., 2001, *Industrial Control Systems Design*, John Wiley, Chichester.

11. Slotine J. J. and W. Li. Applied Nonlinear Control, Prentice-Hall International, 1991.

12. Jukes K A and Grimble, M J, 1981, *A Note on a Compatriot of the Real Marcinkiewicz Space*, Int. J of Control, Vol. 33, No. 1, pp. 187-189.

13. Grimble, M. J., Jukes, K.A. and Goodall, D. P., 1984, *Nonlinear Filters and Operators and the Constant Gain Extended Kalman Filter*, IMA Journal of Mathematical Control and Information, 1, pp 359-386.

14. Hammett, K. D., 1997, *Control of Non-linear Systems via State-feedback State-dependent Riccati Equation Techniques*, PhD Dissertation, AFIT, Dayton, Ohio.

15. Cloutier J.R., 1997, *State-Dependent Riccati Equation Techniques*: *An Overview*, Proc ACC, Albuquerque, June.

16. Sznaier M., J. Cloutier, D. Jacques and C. Mracek, 1998, *A Receding Horizon State Dependent Riccati Equation Approach to Sub Optimal Regulation of Nonlinear Systems*, Proc. 37th IEEE Conference on Decision and Control, Tampa, pp. 1792-1797.

17. Grimble, M. J., 2005, *Robust Industrial Control*, Wiley, Chichester.

18. Genc, A. U., 2002, *Linear Parameter-Varying Modeling and Robust Control of Variable Cam Timing Engines*, PhD thesis, Wolfson College, University of Cambridge.

19. Wei, X. and del Re, L., 2007, *Gain Scheduled* H_∞ *Control for Air Path Systems of Diesel Engines Using LPV Techniques*, IEEE Trans on Control Systems Technology, Vol. 15, No. 3.

20. X. Wei, 2006, *Advanced LPV Techniques for Diesel Engines*, PhD thesis, Johannes Kepler Univ of Linz,Austria.

21. M. Alpbaz, H. Hapo lu and Ç. Güresinli, Application of nonlinear generalized minimum variance control to a tubular flow reactor, Computers and Chemical Engineering, Volume 22, Supplement 1, 15 March 1998, pp. 839-842.

22. H. Khalil, 2002, Nonlinear Systems, 3rd Ed., Prentice Hall.

23. Q.M. Zhu, Z. Ma, and K. Warwick, A neural network enhanced generalized minimum variance self-tuning control for nonlinear discrete-time systems, IEE Proc.-Control Theory Appl., Vol. 146, No. 4, pp. 319-326, 1999.

24. B. Anderson and J. Moore, 1979, Optimal Filtering, Prentice-Hall.

25. Grimble, M. J., and Johnson, M. A., 1988, *Optimal Multivariable Control and Estimation Theory: theory and applications*, Vols. I and II, John Wiley, London.

26. Shamma, J. S. and M. Athans, "Gain scheduling: potential hazards and possible remedies," IEEE Control Syst. Mag., 12(3), pp. 101-107, 1992.
27. Geering, H. P., "Concepts of LPV control with applications in automotive engine control," Proc. of the IEEE Int.' Symp. on Industrial Electronics -- ISIE 2005, pp. 241-246, Dubrovnik, Croatia, June 2005.