New Ideas in Performance Assessment and Benchmarking of Nonlinear Systems

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Presentation Outline

• Introduction
• Review of the linear GMV control theory
• Nonlinear GMV controller
• Performance assessment against NGMV controller
• Simulation example
• Summary
Introduction

• Minimum variance is the most popular stochastic benchmark: simple, meaningful, easy to calculate
• Comparison against the best possible linear controller
• However, all real plants are nonlinear
• Need for high performance control over wide operating range → nonlinear control
• Introduction of NGMV – a new simple nonlinear controller
Minimum Variance Control – a few dates

1970  Åström: MV controller for linear minimum phase plants
1971  Clarke & Hastings-James: Generalized MV criterion
1975  Clarke & Gawthrop: Self-tuning GMV controller
1988  Grimble: GMV control law revisited
1992  Harris, Desborough: MV benchmarking, “Harris” index
1999  Huang & Shah: MIMO MV benchmarking
2002  Grimble: GMV benchmarking
2003  Grimble: Nonlinear GMV control
• Introduction
• **Review of the linear GMV control theory**
• Nonlinear GMV controller
• Performance assessment against NGMV controller
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LTI System Model

\[ W(z^{-1}) = z^{-k} \frac{B(z^{-1})}{A(z^{-1})} \]

\[ W_r(z^{-1}) = \frac{E_r(z^{-1})}{A(z^{-1})} \]

\[ W_d(z^{-1}) = \frac{C_d(z^{-1})}{A(z^{-1})} \]

\[ S = (1 + WC_0)^{-1} \]

- control error
\[ e_t = r_t - y_t = S(z^{-1})(r_t - d_t) \]

- control signal
\[ u_t = C_0(z^{-1})S(z^{-1})(r_t - d_t) \]

- Independent white noise sequences
\[ \xi_t, \zeta_t \]
Spectral Factorisation

Aim: combine all the stochastic inputs into one noise signal

Linear models driven by white noise

\[ f(t) = Y_f(z^{-1})\varepsilon(t) \]

\[ \varepsilon(t) \text{ - zero mean white noise of unit variance} \]

\[ f(t) = r(t) - d(t) \]

\[ \Phi_{ff} = \Phi_{rr} + \Phi_{dd} = W_r W_r^* + W_d W_d^* \]

\[ Y_f Y_f^* = \Phi_{ff} \]

spectral factor
Minimum Variance Control

To minimize: output variance  \[ J_{MV} = E[y^2(t)] \]

\[ y(t) = Wu(t-k) + Y_f \varepsilon(t) \]

\[ = F \varepsilon(t) + Wu(t-k) + R \varepsilon(t-k) \]

Optimal control:  \[ u^{MV}(t) = -\frac{R}{W} \varepsilon(t) = -\frac{R}{WF} y(t) \]

MV controller works when:
• the plant \( W \) is invertible (minimum-phase)
• reference and disturbance models are representative of the actual signals entering the system

\[ Y_f = F + z^{-k} R \]

Diophantine equation
Generalised Minimum Variance Criterion

To minimize: variance of the “generalized” output $\phi_0(t)$:

$$J_{GMV} = E[\phi_0^2(t)]$$

$$\phi_0(t) = P_c e(t) + F_c u(t)$$

Dynamic weightings

$$P_c = \frac{P_{en}}{P_{cd}}, \quad F_c = z^{-k} \frac{F_{ck}}{F_{cd}}$$

$e(t)$ affected by $u(t-k)$
GMV problem can be recast as an MV problem for the “generalized” plant:

\[ \phi(t) = P_c (-z^{-k} W_k u(t) + Y_f \varepsilon(t)) + F_c u(t) \]

\[ = z^{-k} (F_{ck} - P_c W_k) u(t) + P_c Y_f \varepsilon(t) \]
GMV Controller

MV control of the plant \((P_c W - F_c)\):

\[
\phi_0(t) = (P_c W - F_c)u(t) + P_c Y_f \varepsilon(t)
\]

\[
= F \varepsilon(t) + (P_c W - F_c)u(t - k) + R \varepsilon(t - k)
\]

Diophantine equation

\[
P_c Y_f = F + z^{-k} R
\]

statistically independent terms

Optimal GMV control:

\[
u^{GMV}(t) = -\frac{R}{(P_c W - F_c)F} \phi_0(t)
\]

Polynomial solution:

\[
u^{GMV}(t) = -\frac{R}{(F_{ck} - FY_f^{-1}W_k)Y_f} e(t) = \frac{GF_{cd}}{HP_{cd}} e(t)
\]
GMV Controller Implementation

Subsystems depending only on the noise models, error weighting and the time delay $k$

Delay-free plant model
• Introduction
• Review of the linear GMV control theory
• Nonlinear GMV controller
• Performance assessment against NGMV controller
• Simulation example
• Summary
**Nonlinear System Description**

**Nonlinear plant model:**
\[
(Wu)(t) = z^{-k}(W_ku)(t)
\]

**Disturbance model:** (assumed linear)
\[
W_d = A_f^{-1}C_d
\]

**Reference model:** (assumed linear)
\[
W_r = A_f^{-1}E_r
\]
Nonlinear plant model may be given in a very general form, e.g.:  
• state-space formulation  
• neural network / neuro-fuzzy model  
• look-up table  
• Fortran/C code  
It can include both linear and nonlinear components, e.g. Hammerstein model:

Just need to obtain the output to given input signal
Nonlinear GMV Problem Formulation

The cost function is as in the linear case:

\[
J_{NGMV} = E[\phi_0^2(t)]
\]

with \( \phi_0(t) = P_c e(t) + (F_c u)(t) \)

\( P_c = \frac{P_{cn}}{P_{cd}} \) - linear error weighting

\( (F_c u)(t) = z^{-k} (F_{ck} u)(t) \) - possibly nonlinear control weighting

• Control weighting invertible and potentially nonlinear to compensate for plant nonlinearities
• The weighting selection is restricted
Nonlinear GMV Problem Solution

The approach also similar:

$$\phi(t) = P_c (-z^{-k} (W_k u)(t) + Y_f \varepsilon(t)) + (F_c u)(t)$$

$$= z^{-k} (F_{ck} - P_c W_k) u(t) + P_c Y_f \varepsilon(t)$$

$$\phi(t) = F \varepsilon(t) + \left[ (F_{ck} - P_c W_k) u(t - k) + R \varepsilon(t - k) \right]$$

$$\varepsilon(t) - \text{white noise (sequence of independent random variables)}$$

$$P_c Y_f = F + z^{-k} R$$

Diophantine equation

Optimal control:

$$u^{NGMV}(t) = - (F_{ck} - P_c W_k)^{-1} R \varepsilon(t)$$

stable causal nonlinear operator inverse
Nonlinear Operator and its Inverse

By definition:

\[
(P_c W_k - F_{kk})u = P_c (W_k u)(t) - (F_{kk} u)(t) = \psi(t)
\]

\[
u(t) = F_{kk}^{-1} \left[ P_c (W_k u)(t) - \psi(t) \right]
\]

- Control weighting assumed invertible
- For the closed-loop stability, the nonlinear operator must be invertible in the operating region
- Problem: algebraic loop
Algebraic Loop

Problem solutions:
• solve the loop iteratively on-line
• introduce an additional delay in the loop
• transform into equivalent problem:

→ split the nonlinear operator into two parts involving a delay-free
term $N_0$ and a term that depends upon past values of the control
action $N_1$:

$$\psi(t) = (P_c W_k - F_{ck})u = (N_0 u)(t) + (N_1 u)(t)$$

$$u(t) = N_0^{-1}(\psi(t) - (N_1 u)(t))$$

$$N_0 = N \bigg|_{z^{-1}=0} = (P_c W_k - F_{ck}) \bigg|_{z^{-1}=0}$$

$$N_1 = N - N_0.$$
Controller Implementation

\[ u^{NGMV}(t) = -[(F_{ck} - FY_f^{-1}W_k)^{-1}RY_f^{-1}e](t) \]
Bias and Steady-State Levels

- So far the assumption was on zero mean exogenous signals
- Behaviour at an operating point of interest

Signal notation: \( x = \bar{x} + \mathcal{F}_{\epsilon} \)

Modified signal \( \dot{\phi}_0(t) \):
\[
\dot{\phi}_0(t) = P_c e(t) + (F_c u)(t) - (F_c \bar{u})(t)
\]

Re-derived control:
\[
u^{opt}(t) = -[(F_{ck} - F Y_f^{-1} W_k)^{-1} R Y_f^{-1} e](t) + \bar{u}
\]
Design of the Weightings

- Restriction on the choice of weightings: invertibility of the nonlinear operator
  \[ (P_c W_k - F_{kk}) \]
- Control weighting may be used to control a non-invertible plant
- Admissible and meaningful choice of the weightings is the subject of current research
- Adaptation of the linear-case rules of thumb:
  - \( P_c \) normally high at low frequencies to guarantee integral action
  - \( F_c \) high at large frequencies to provide sufficient controller roll-off
  - properties close to those of LQG-type controllers
  - closed-loop bandwidth normally close to cross-over frequency of the open-loop system \( \rightarrow \) loop shaping
Design of the Weightings (cont.)

Consider $\Phi_{ck}$ linear and negative: $F_{ck} = -F_k$

Then

$$\left( P_c W_k + F_k \right) u = F_k \left[ 1 + \frac{P_c}{F_k} W_k \right] u$$

return-difference operator for a feedback system with the delay-free plant and controller $\frac{P_c}{F_k}$

Consider the delay-free plant $\Omega_k$ and assume a PID controller $K_{PID}$ exists to stabilize the closed-loop system.

Then a starting point for the weighting choice that will ensure the operator $\left( P_c W_k + F_k \right)$ is stably invertible is

$$\frac{P_c}{F_k} = K_{PID}$$
Special Case: Nonlinear MV Controller

Note that in the limiting case, for a square system, when $F_{ck} \to 0$ the optimal control signal becomes

$$u_{mv}(t) = (P_c W_k)^{-1} R Y_f^{-1} \left( r(t) - d(t) \right)$$

Clearly the minimum variance control for the nonlinear system includes the stable inverse of the plant model, when one exists.
Special Case: Actuator Saturation

Problem: select invertible control weighting $\Phi_{ck}$ such that an anti-windup mechanism is achieved
Actuator Saturation (cont.)

Control weighting choice:

\[
(F_{ck}u)(t) = (\hat{F}_{ck}u)(t) + \frac{\rho}{1-z^{-1}}[u - f(u)]
\]

nominal weighting

integrator, becomes active at saturation

Control signal becomes:

\[
\hat{u}(t) = F_{ck}^{-1}\left(F_0Y_f^{-1}(W_ku)(t) - RY_f^{-1}\epsilon(t) - \frac{\rho}{1-z^{-1}}(u - f(u))\right)
\]
Actuator Saturation – Block Diagram

anti-windup scheme

Plant model without delay
Actuator Saturation - Example

integral windup

saturation level
Nonlinear Smith Predictor

• Optimal NGMV controller can be expressed in a similar form to that of a Smith predictor
• Introduction of this structure limits the application of the solution on open-loop unstable systems
• The structure is intuitively reasonable and should be valuable in applications
• The Smith predictor results by rearranging the controller structure
The control loop can be rearranged as follows:
Nonlinear Smith Predictor

When the error weighting $P_c$ includes an integrator, it must be placed in the inner error channel:
Extensions

• NGMV Feedback, Feedforward and Tracking control
• Multivariable version → time-delay matrix
• Modelling issues – “Neuro-fuzzy NGMV control”
• State-space representation
• Introduction
• Review of the linear GMV control theory
• Nonlinear GMV controller
• **Performance assessment against NGMV controller**
• Simulation example
• Summary
Estimation of the Minimum Variance

NGMV controller cancels the (generalized) plant dynamics and the generalized output signal \( \phi_0(t) \) is a moving-average time series:

\[
\phi_0^{ngmv}(t) = F \varepsilon(t) = f_0 \varepsilon(t) + f_1 \varepsilon(t-1) + ... + f_{k-1} \varepsilon(t-k+1)
\]

The minimum variance (the benchmark) follows as

\[
J_{\text{min}} = \text{Var}[F \varepsilon(t)] = \sum_{i=0}^{k-1} f_i^2
\]

This value depends only on the noise model and the plant time delay.

**Problem:** estimate \( J_{\text{min}} \) from the collected closed-loop data.
“Harris Algorithm”

Harris (1989) and Desborough and Harris (1992, 1993):
• model the controller-dependent part of the output as AR time series
• estimate the minimum variance as the residual error variance

The generalized algorithm applies to the signal $\phi_0(t)$ rather than $y(t)$:

$$
\phi_0(t) = F \varepsilon(t) + \sum_{i=0}^{m} \alpha_i \phi_0(t-k-i)
$$

Collect N samples of data and write in matrix form:

$$
\bar{\phi} = X \bar{\alpha} + \bar{\varepsilon} \quad \rightarrow \quad \bar{\alpha} = (X^T X)^{-1} X^T \bar{\phi}
$$

Minimum variance estimate:

$$
\hat{\sigma}_{mv}^2 = \frac{1}{N-k-2m+1} (\bar{\phi} - X \bar{\alpha})^T (\bar{\phi} - X \bar{\alpha})
$$
FCOR Algorithm

Huang and Shah (1999): Filtering and Correlation algorithm

- model the output as an AR time series and estimate the white noise generating sequence (innovations sequence)
- correlate the obtained white noise with the output

As applied to the generalized signal $\phi_0(t)$:

**Whitening process:**
$$\phi_t = \frac{1}{A(q^{-1})} \varepsilon_t \quad \Rightarrow \quad \varepsilon_t = A(q^{-1})\phi_t$$

**Cross-correlation:**
$$r_{\phi \varepsilon}(0) = E[\phi_t \varepsilon_i] = f_0$$
$$r_{\phi \varepsilon}(1) = E[\phi_t \varepsilon_{i-1}] = f_1$$
$$\vdots$$
$$r_{\phi \varepsilon}(k-1) = E[\phi_t \varepsilon_{i-k+1}] = f_{k-1}$$

**Minimum variance:**
$$J_{\text{min}} = \sum_{i=0}^{\infty} f_i^2$$
Controller Performance Index

For the existing controller

\[ J = \text{Var} [\phi_0] \geq J_{\text{min}} \]

Definition of the Controller Performance Index:

\[ \kappa = \frac{J_{\text{min}}}{J} \]

- 1 \hspace{1cm} (NGMV control)
- 0 \hspace{1cm} (very poor control)

"Harris index"
• Introduction
• Review of the linear GMV control theory
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  • Simulation example
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Continuous Stirred Tank Reactor

**Energy balance equation**

\[
\mathcal{E}(t) = \frac{q(t)}{V} (T_0 - T(t)) + \frac{\Delta H}{\rho C_p} k_0 C_a(t) \exp\left(-\frac{E}{R T(t)}\right) \\
+ \frac{\rho_c C_{pc}}{\rho C_p V} \left\{ q_c(t) \left[ 1 - \exp\left(-\frac{-hA}{\rho_c C_{pc} q_c(t)}\right) \right] (T_{co} - T(t)) \right. \\
\]

**Material balance equation**

\[
C_a(t) = \frac{q(t)}{V} (C_{a0} - C_a(t)) - k_0 C_a(t) \exp\left(-\frac{E}{R T(t)}\right) \\
\]

**Table of Symbols**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Unit</th>
<th>Nominal Value</th>
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<tbody>
<tr>
<td>Product concen.</td>
<td>(C_a)</td>
<td>mol/l</td>
</tr>
<tr>
<td>Reactor temp.</td>
<td>(T)</td>
<td>K</td>
</tr>
<tr>
<td>Coolant flow rate</td>
<td>(q_c)</td>
<td>l/min</td>
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<td>Process flow rate</td>
<td>(q)</td>
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<td>Feed concen.</td>
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<td>Feed temp.</td>
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<td>Inlet coolant temp.</td>
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<td>CSTR volume</td>
<td>(V)</td>
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<td>Heat transfer term</td>
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<tr>
<td>Reaction rate constant</td>
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<td>Activation energy term</td>
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<tr>
<td>Coolant specific heat</td>
<td>(C_{pc})</td>
<td>cal/g/K</td>
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</table>
Open-Loop Step Responses

(a) Linear model
(b) Nonlinear model

Linearised model

\[ W_{lin} = z^{-5} \frac{10^{-4}(0.01571z^{-1} + 1.863z^{-2} + 0.4614z^{-3} - 1.14z^{-3})}{1 - 2.338z^{-1} + 1.88z^{-2} - 0.5098z^{-3}} \]
Optimal Controller Design

Disturbance model:

\[ W_d = \frac{0.001}{1 - 0.95z^{-1}} \]

Dynamic weightings:

\[ P_c = \frac{1 - 0.85z^{-1}}{1 - z^{-1}} \]

\[ F_{ck} = -0.015(1 - 0.1z^{-1}) \]
GMV Controller Design

Sensitivity function plots for the PID and the linear GMV controller

Bode Diagram

- Magnitude (dB)
- Frequency (rad/sec)
## Stochastic Performance

<table>
<thead>
<tr>
<th>$C_a$ [mol/l]</th>
<th>Controller</th>
<th>Var[$C_a$]</th>
<th>Var[$q_c$]</th>
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### Benchmarking Results

#### Benchmarking results (nominal operating point)

<table>
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<th>Controller</th>
<th>Method</th>
<th>Jmin</th>
<th>J</th>
<th>eta</th>
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<td>Harris</td>
<td>7.301e-2</td>
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<td>7.320e-2</td>
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<td>FCOR</td>
<td>7.304e-2</td>
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<td>7.322e-2</td>
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#### Benchmarking results (operating point $C_a=0.06 \text{ mol/l}$)

<table>
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#### Benchmarking results (operating point $C_a=0.12 \text{ mol/l}$)

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<td>7.241e-2</td>
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</table>
Comments on the Results

• NGMV shows best performance over the whole operating range
• While the linear GMV control adequate for small deviations from the nominal operating point, its performance degrades when away from it
• Overall performance of the existing PI controller remains relatively unchanged compared with the linear GMV controller – this is due to greater robustness of this simple controller
• Transient performance comparable; NGMV more robust than linear GMV away from the operating point
• Introduction
• Review of the linear GMV control theory
• Nonlinear GMV controller
• Performance assessment against NGMV controller
• Simulation example

• **Summary**
Summary

• Simple nonlinear control algorithm introduced
• Generalization of the established linear GMV controller
• Anti-windup mechanism easily incorporated
• Can be represented in the “nonlinear Smith predictor” form
• Candidate for a benchmark controller